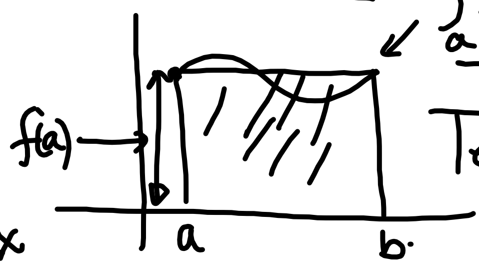
Deterministic Rules.

$$\int_a^b f(x)\,dx \approx f(a)(b-a)$$

A Bad Rule.



Test functions. $1, x, x^2, \dots$

$f(x)=1$: $\int_a^b f(x)\,dx = \int_a^b 1\,dx = b-a.$ | $f(a)(b-a) = b-a$ ✓

$f(x)=x$ $\int_a^b x\,dx = \frac{x^2}{2}\Big|_{x=a}^{x=b} = \frac{b^2-a^2}{2}$ | $a(b-a)$ ✗

Error: $\int_a^b f(x)\,dx - f(a)(b-a)$.

Trick: $\int_a^b \underset{u'}{1}\cdot \underset{v}{f(x)}\,dx = \underset{u}{(x-b)}\,\underset{v}{f(x)}\Big|_{x=a}^{x=b} - \int_a^b \underset{u}{(x-b)}\,\underset{v'}{f'(x)}\,dx$

$u' = 1$

$u = x + C$

choose $C = -b$

$u = x - b$

$$= \{0 - (a-b)f(a)\} + \int_a^b (b-x)f'(x)\,dx$$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{true}} = \underbrace{f(a)(b-a)}_{\substack{\text{estimate}\\ \text{Rule}}} + \underbrace{\int_a^b (b-x)f'(x)\,dx}_{\text{error}}.$$

$$E(f) = \int_a^b (b-x) f'(x)\,dx.$$

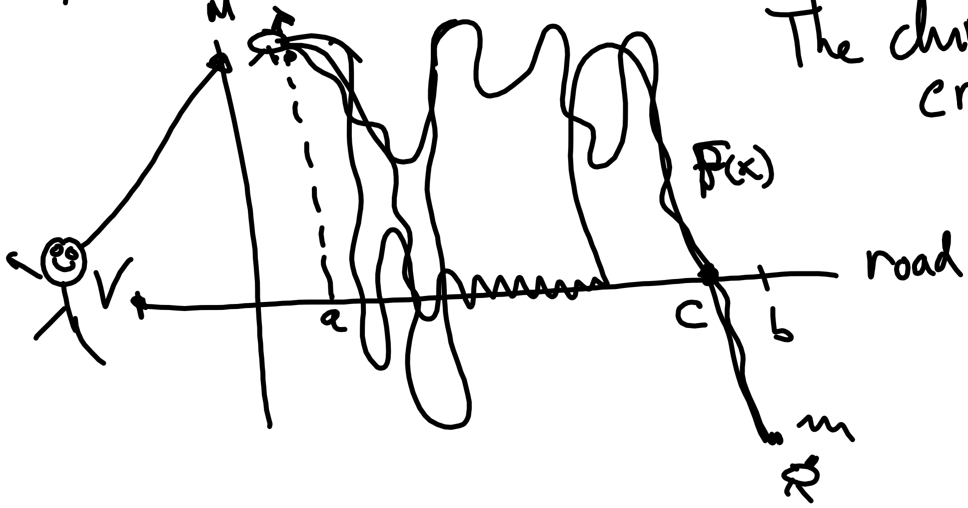
$$G(x) = b - x$$

$$G(x)\,F(x) \qquad \underline{m} \le F(x) \le M \qquad a \le x \le b$$

Bolzano's Theorem. $\qquad G(x) \ge 0.$

$$m = \frac{\int_a^b G(x)\, m\, dx}{\int_a^b G(x)\,dx} \le \frac{\int_a^b G(x) \boxed{F(x)}\, dx}{\int_a^b G(x)\, dx} \le \frac{\int_a^b G(x)\, M\, dx}{\int_a^b G(x)\, dx} = M \frac{\cancel{\int_a^b G}}{\cancel{\int_a^b G}}$$

The chicken and the Road Theorem.



The chicken crossed the road!

$\underline{f(c) = V}$

$$m \leq \frac{\int_a^b F(x)\,G(x)\,dx}{\int_a^b G(x)\,dx} \leq M$$

$= F(c)$

$\exists\ c \in [a,b]$ so that

$$\Rightarrow \int_a^b F(x)\,G(x)\,dx = F(c)\int_a^b G(x)\,dx$$

$$E(f) = \int_a^b \underbrace{(b-x)}_{G(x)}\,\underbrace{f'(x)}_{F(x)}\,dx$$

$$= f'(c)\int_a^b (b-x)\,dx = f'(c)\left\{-\frac{(b-x)^2}{2}\right\}_{x=a}^{x=b}$$

$$= f'(c)\left\{0 + \frac{(b-a)^2}{2}\right\}.$$

$$= f'(c)\,\frac{(b-a)^2}{2}$$

$$\int_a^b f(x)\,dx = f(a)(b-a) + \frac{(b-a)^2}{2} f'(c)$$

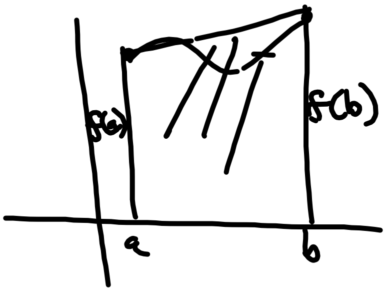
integral estimate error

for some c, $a \le c \le b$

$$\int_a^b f(x)\,dx = f(b)(b-a) - \frac{(b-a)^2}{2} f'(c_1)$$

Average: $\int_a^b f(x)\,dx = (b-a)\left(\frac{f(a)+f(b)}{2}\right) + \cdots ?$

Trapezoidal Rule



$$A_T = (b-a)\frac{(f(a)+f(b))}{2}$$