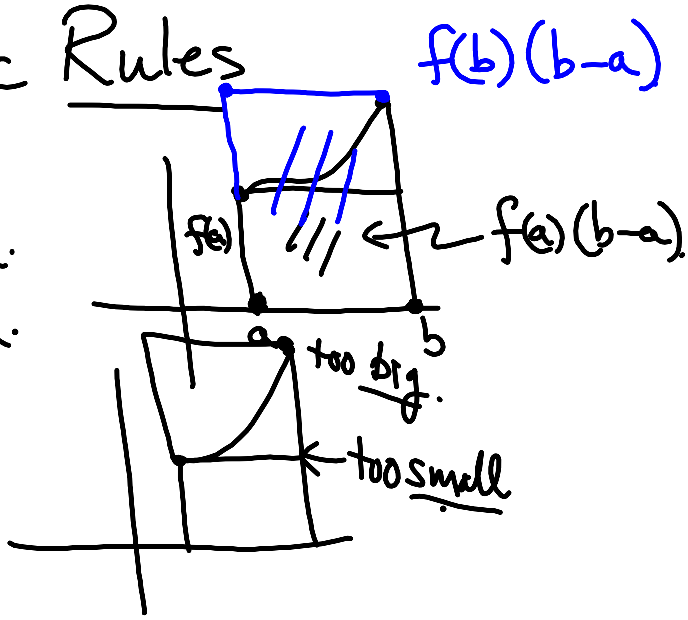
Deterministic Rules

Two bad rules.

1) Left endpoint rule.
2) Right endpoint rule.
3) Average should be **better**!



<u>Errors</u>:

$$\int_a^b f(x)\,dx = \int_a^b \underset{\substack{\uparrow \\ u'=1 \\ u=x-b.}}{1} \times \underset{v}{f(x)}\,dx$$

$$= \underset{u}{(x-b)}\,\underset{v}{f(x)}\Big|_{x=a}^{b} - \int_a^b \underset{u}{(x-b)}\,\underset{v'}{f'(x)}\,dx.$$

$$= \{0-(a-b)f(a)\} + \int_a^b (b-x)f'(x)\,dx$$

$$= f(a)(b-a) \leftarrow \text{L.E.R.}$$

$$+ \int_a^b \underbrace{\underset{\substack{\uparrow \\ G(x)}}{(b-x)}\,\overset{F(x)}{f'(x)}\,dx}_{\text{error}}$$

$$\text{Error} = \int_a^b \underbrace{(b-x)}_{G(x)} \underbrace{f'(x)}_{F(x)} dx = f'(c) \int_a^b (b-x)dx$$

$$F(c) \int_a^b G(x)dx$$

$$= f'(c) \left\{ -\frac{(b-x)^2}{2} \Big|_{x=a}^{x=b} \right\}.$$

$$= f'(c)\frac{(b-a)^2}{2}.$$

<u>Rule</u>: $\int_a^b f(x)dx = f(a)(b-a) + f'(c)\frac{(b-a)^2}{2}$

Right Endpoint Rule

$f(b)(b-a)$

$$\int_a^b f(x)\,dx = f(a)(b-a) + f'(c)\frac{(b-a)^2}{2}$$

$$\int_a^b f(x)\,dx = \int_a^b \underset{u'}{1}\,\underset{v}{f(x)}\,dx$$

$u = x-a$

$$= \underset{u}{(x-a)}\,\underset{v}{f(x)}\Big|_{x=a}^{x=b} - \int_a^b \underset{u}{(x-a)}\,\underset{v'}{f'(x)}\,dx$$

$$= f(b)(b-a) - f'(c)\int_a^b (x-a)\,dx$$

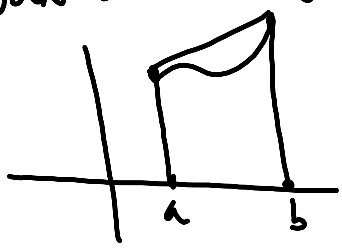
$$= f(b)(b-a) - f'(c)\frac{(b-a)^2}{2}$$

not the same

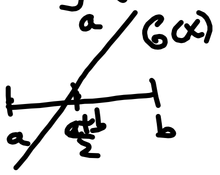
$$\int_a^b f(x)dx = f(a)(b-a) + f'(c_1)\frac{(b-a)^2}{2}$$

$$\int_a^b f(x)dx = f(b)(b-a) - f'(c_2)\frac{(b-a)^2}{2}$$

Average: $\displaystyle\int_a^b f(x)dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right)$



$$\int_a^b f(x)\,dx = \int_a^b \underset{u'}{1}\, \underset{v}{f(x)}\,dx = \overset{u}{\left(x-\frac{a+b}{2}\right)} \overset{v}{f(x)}\Big|_{x=a}^{b}$$



$$u = x-\frac{a+b}{2}$$

$$-\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx$$

$$= \left\{\left(b-\frac{a+b}{2}\right) f(b) - \left(a-\frac{a+b}{2}\right) f(a)\right\}$$

$$-\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx$$

$$= \frac{b-a}{2} f(b) + \frac{b-a}{2} f(a) - \overbrace{\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx}^{\text{error}}$$

$$= (b-a)\left(f(a)+f(b)\right)/2$$

Error: $-\int_a^b \underbrace{\left(x-\frac{a+b}{2}\right)}_{u'} \underbrace{f'(x)}_{v}\,dx = \quad u \quad v$

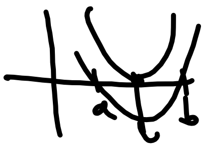
$\left(x-\frac{a+b}{2}\right)^2 + c'$

$u' = x - \frac{a+b}{2}$

$u = \frac{x^2}{2} - \frac{a+b}{2}x + C$

$0 = \frac{a^2}{2} - \left(\frac{a+b}{2}\right)a + C$ at a

$\iff C = \frac{(a+b-a)a}{2} = \frac{ab}{2}$



$\frac{b^2}{2} - \frac{a+b}{2}b + C = 0$

$C = \frac{(a+b)b - b^2}{2}$

$= \frac{\left((a+\cancel{b}) - \cancel{b}\right)b}{2}$

$= \frac{ab}{2}$

A minor miracle

$$u = \frac{x^2}{2} - \frac{a+b}{2}x + \frac{ab}{2} = \frac{(x-a)(x-b)}{2}$$

$$\int_a^b \underbrace{\left(x - \frac{a+b}{2}\right)}_{u'} \underbrace{f'(x)}_{v} dx = \underbrace{\frac{(x-a)(x-b)}{2}}_{u} \underbrace{f'(x)}_{v}\Big|_{x=a}^{x=b} - \int_a^b \underbrace{\frac{(x-a)(x-b)}{2}}_{u} \underbrace{f''(x)}_{v'} dx$$

$$u = \frac{(x-a)(x-b)}{2} \qquad = 0 + \frac{1}{2}\int_a^b \underbrace{(x-a)(b-x)}_{G(x)} \underbrace{f''(x)}_{F(x)} dx$$

$$\text{Error} := -\frac{1}{2} f''(c) \int_a^b (x-a)(b-x)\,dx$$

$$= -\frac{1}{2} f''(c) \frac{(b-a)^3}{6} = -\frac{1}{12} \underbrace{f''(c)}_{=0 \text{ if } f(x) = Ax+B} (b-a)^3$$

Shortcut

Look at the polynomials for which the formula is exact!

$$\frac{b-a}{2}\left(f(a)+f(b)\right)$$

Estimate

$f(x)=1$: $\frac{b-a}{2}(1+1)=b-a$ ✓

$f(x)=x$ $\frac{(b-a)}{2}(a+b)$ ✓

$= \frac{b^2-a^2}{2}$

$f(x)=x^2$ $\frac{b-a}{2}(a^2+b^2)$

$$\int_a^b f(x)dx$$

$$\int_a^b 1\,dx = x\Big|_a^b = b-a$$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$$

$$\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3}$$

$$\frac{(\cancel{b-a})}{2}(a^2+b^2) \overset{?}{\neq} \frac{b^3-a^3}{3} = \frac{(\cancel{b-a})}{3}(b^2+ab+a^2)$$

Rule is exact for polys. of degree 1!

$$E = (\underline{\underline{\text{const}}}) \times f^{(2)}(c)$$

Take $f(x) = \underline{x^2}$

$$E = \int_a^b f(x)\,dx - \frac{b-a}{2}(f(a)+f(b)) = \not\ 0$$

$$\therefore\ X = \frac{1}{2}\left\{ \int_a^b x^2 dx - \frac{b-a}{2}\left(a^2+b^2\right) \right\}$$

$$= \frac{1}{2}\left\{ \frac{b^3-a^3}{3} - \frac{b-a}{2}\left(a^2+b^2\right) \right\} \to -\frac{1}{12}(b-a)^3$$

$$= \frac{1}{2}(b-a)\left\{ \frac{b^2+ab+a^2}{3} - \frac{a^2+b^2}{2} \right\}$$

$$= \frac{1}{2}(b-a)\left\{ \underline{2b^2+2ab+2a^2-3a^2-3b^2} \right\}$$

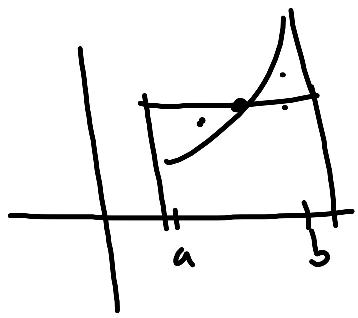
$$= \frac{1}{12}(b-a)\left\{ -b^2+2ab\ \underline{6}\ a^2 \right\} = -\frac{1}{12}(b-a)\left(b^2-2ab+a^2\right)$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right) \boxed{-\frac{(b-a)^3}{12} f''(c)}$$

Adaptive Quadrature

Midpoint Rule: $f\left(\frac{a+b}{2}\right)(b-a)$



Rule $(b-a)f\left(\frac{a+b}{2}\right)$ $\qquad \int_a^b f(x)dx$

$f(x)=1$: $(b-a)x$: ✓ $\qquad \int_a^b 1\,dx = b-a$

$f(x)=x$ $\quad (b-a)\frac{b+a}{2}$ ✓ $\qquad \int_a^b x\,dx = \frac{x^2}{2}\Big|_{x=a}^{b} = \frac{b^2-a^2}{2}$

$f(x)=x^2$ $\quad (b-a)\left(\frac{a+b}{2}\right)^2$ X $\qquad \int_a^b x^2 dx = \frac{b^3-a^3}{3}$

$$E = X \cdot \frac{f^{(2)}(c)}{2} \Rightarrow X = \frac{(b-a)^3}{24} \Rightarrow E = \frac{(b-a)^3}{24} f^{(2)}(c)$$

$f(x)=x^2$

$$E_T = -\frac{(b-a)^3}{12} f''(c_1)$$

$$E_M = +\frac{(b-a)^3}{24} f''(c_2)$$

$\frac{2M+T}{3}$ should be better!

$$\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right\}(b-a)$$

$$\int = \frac{b-a}{6}\left\{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right\}$$

Cavalieri + Simpson.

$$S = \frac{b-a}{6}\left\{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right\} \qquad \int_a^b f(x)\,dx$$

$\underline{f=1}$ ✓ $\quad f=x$ ✓ $\quad f=x^2$

$$\frac{b-a}{6}\left(a^2+4\left(\frac{a+b}{2}\right)^2+b^2\right) \overset{\checkmark}{=\!=} \frac{b^3-a^3}{3}$$

$$\underline{f=x^3} \quad \frac{b-a}{6}\left\{a^3+4\left(\frac{a+b}{2}\right)^3+b^3\right\} \overset{\checkmark}{=\!=} \frac{b^4-a^4}{4}$$

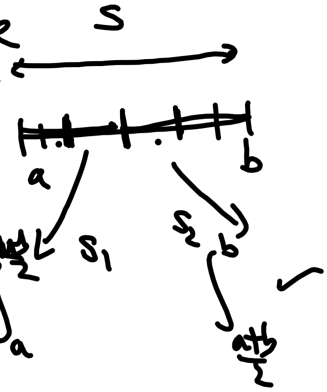
$\underline{f=x^4}$ X $\qquad E_S = X \times f^{(4)}(c)$

$4 \times 3 \times 2 \times 1$

$$\boxed{E_S = -\frac{\left(\frac{b-a}{2}\right)^5}{90} f^{(4)}(c)}$$

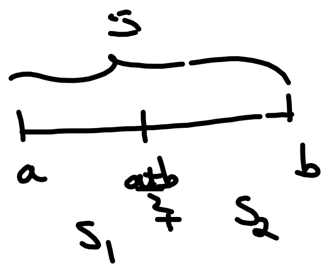
small if the interval is small!

Adaptive Quadrature



Estimate:

$$\left|\int_a^b f(x)dx - (S_1+S_2)\right| \approx \left|\frac{S-(S_1+S_2)}{15}\right|$$



Two different estimates for $\int_a^b f(x)dx$

1. S
2. $S_1 + S_2$