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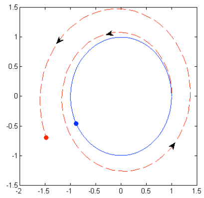

A Simple Numerical Model of an Orbit

by Len Bos [*]

1. The Model

Numerical models are often used to try to understand complex physical behaviour. For example, much of the debate about Global Warming is with regard to predictions made by sophisticated climate models. But how reliable are such models? Do they really capture the physical reality of a phenomenon? A basic problem is that we won't know for sure till afterwards whether or not a model gave correct predictions. Here, I would like to present a very simple example of a numerical model for a very simple problem where we know the exact solution and can compute exactly what the model predicts and then compare the two. I hope that you find it interesting!

The example is this. Consider a "satellite'' in a purely circular orbit with coordinates at any given time given by $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Here we are starting with the solution, but normally in a science setting we only know certain equations that describe only implicitly what it is we are studying. Typically these are differential equations of some type and number. In our simple example these would be

$$\frac{dx}{dt} = -\sin(t) = -y \quad , \quad \frac{dy}{dt} = \cos(t) = x \qquad (1)$$

with initial conditions $x(0) = 1$ and $y(0) = 0$ so let's pretend that this system (1) is what we were given and what we solved to find our formulas for $x(t)$ and $y(t)$. It is convenient to rewrite (1) in terms of matrices and vectors as

$$\frac{d}{dt}\vec{x} = A\vec{x} \qquad (2)$$

where the vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now for the numerical model. Actually it will be the simplest possible numerical method for solving the system (2). It is called Euler's method and goes as follows. Consider the general first order differential equation, $y'(t) = F(y(t), t)$. From the basic Calculus formula

$$y(t_n) - y(t_{n-1}) = \int_{t_{n-1}}^{t_n} y'(t)\,dt$$

we get that

$$y(t_n) = y(t_{n-1}) + \int_{t_{n-1}}^{t_n} y'(t)\,dt.$$

Euler's method is to estimate the above integral by the area of a rectangle with base $t_n - t_{n-1}$ and height $y'(t_{n-1})$ i.e.,

$$y(t_n) \approx y(t_{n-1}) + (t_n - t_{n-1})y'(t_{n-1}).$$

If we how take equally spaced time steps of size $t_n - t_{n-1} = h$ we end up with the iteration

$$y_n = y_{n-1} + hy'(t_{n-1}) = y_{n-1} + hF(y_{n-1}, t_{n-1}).$$

For our system (1) (or (2)) this results in the iteration

$$x_n = x_{n-1} - hy_{n-1} \quad , \quad y_n = y_{n-1} + hx_{n-1}. \qquad (3)$$

In Figure 1 below we show what happens (for $h = 0.1$). The blue curve is the true orbit and the blue dot is the true position of the satellite at time $t = 10$. The red curve and the red dot are the results of the iteration (3). It doesn't work too well! Actually, if we chose a smaller stepsize h the two positions would be much closer together, but there will always be a fundamental difference – the true solution is a *circular* orbit while the computed solution is a *spiral*!

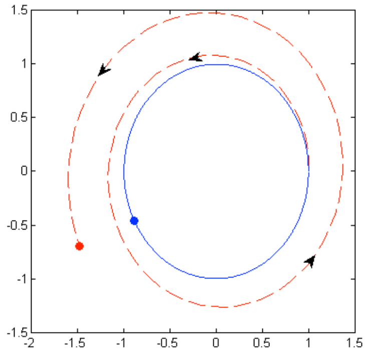

Figure 1

The numerical solution is qualitatively different. It doesn't capture the **true** physical behaviour of the orbit. No matter how small a stepsize you use, after a certain amount of time the computed solution will spiral off into deep space!

In this particular case it is easy to see exactly what is going on with the numerical solution. Let's try to figure this out. First let's express the iteration (3) in terms of vectors:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} + h\begin{bmatrix} -y_{n-1} \\ x_{n-1} \end{bmatrix}. \qquad (4)$$

Since the vector $\langle -y_{n-1}, x_{n-1}\rangle$ is orthogonal to the vector $\langle x_{n-1}, y_{n-1}\rangle$ it is *tangent* to the circle centred at the origin passing through the point (x_{n-1}, y_{n-1}). Hence, geometrically, (4) means that you get to the nth point from the $(n-1)$st point by moving in the direction of the tangent vector to the circle on which the $(n-1)$st point lies, for a distance multiplied by a factor of h. This is illustrated in Figure 2.

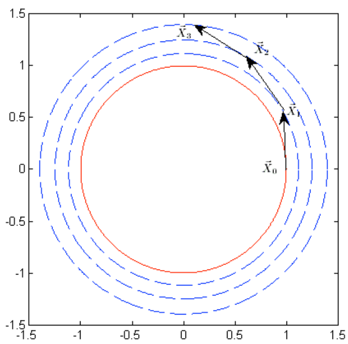

Figure 2

We can even calculate the radii of the various circles. Indeed, if we set

$$R_n = \sqrt{x_n^2 + y_n^2},$$

then by substituting in the formulas of (3), we see that

$R_n^2 = (1+h^2)R_{n-1}^2$,
after a bit of algebra. Since $R_0 = 1$, we then have that
$R_n = (1+h^2)^{n/2}$.
It's now pretty clear why we get a spiral! However, if we look at the equations more closely, we can get even more information. Let's now write the iteration (3) (or (4)) in terms of *matrices* and vectors. This results in

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -h \\ h & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}. \qquad (5)$$

We first re-express the matrix as

$$\begin{bmatrix} 1 & -h \\ h & 1 \end{bmatrix} = \sqrt{1+h^2} \begin{bmatrix} \frac{1}{\sqrt{1+h^2}} & \frac{-h}{\sqrt{1+h^2}} \\ \frac{h}{\sqrt{1+h^2}} & \frac{1}{\sqrt{1+h^2}} \end{bmatrix}.$$

Then notice that since
$(1/\sqrt{1+h^2})^2 + (-h/\sqrt{1+h^2})^2 = 1$,
We may write $1/\sqrt{1+h^2} = \cos(\varphi)$ and $h/\sqrt{1+h^2} = \sin(\varphi)$
for some angle φ (actually the exact value of φ is
$\varphi = \cos^{-1}\left(1/\sqrt{1+h^2}\right))$
so that (5) becomes

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \sqrt{1+h^2}\, R(\varphi) \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}$$

where $R(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$ is the rotation matrix by an angle φ (it's easy to check that $R(\varphi)^n = R(n\varphi)$ -- try it!). Hence we see from (5) that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (1+h^2)^{n/2} \begin{bmatrix} \cos(n\phi) \\ \sin(n\phi) \end{bmatrix}.$$

Comparing this with the exact solution at time $t_n = nh$,

$$\begin{bmatrix} x(t_n) \\ y(t_n) \end{bmatrix} = \begin{bmatrix} \cos(nh) \\ \sin(nh) \end{bmatrix},$$

we can see exactly how much error there is and why!

2. The Gauss-Seidel Correction

If you wanted to calculate the iterates (4) then you would have to do them one at a time – first x_n (usually) and then y_n. Gauss had a clever idea in this regard – if we've already calculated a new value for x_n why not use **it** in the update for y_n instead of the old value x_{n-1}? Figure 3 shows what happens.

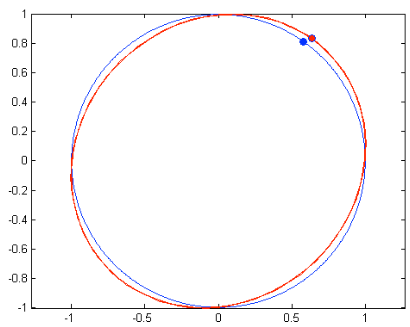

Figure 3

A little explanation is required. The blue dot is the exact solution and the red the computed approximate solution. We used a rather large value of $h = 0.2$ (in order to be able to see a difference) and the positions are after 100 time steps. Hence the total elpased time is $100 \times 0.2 = 20$ units, which corresponds to just after 3 complete orbits. The two positions are really quite close for such a large h!
The approximate solution seems to stay in a bounded orbit, but just not quite the perfect circle that it should be! What actually is this approximate orbit then? We can figure it out! The iteration (4), with the Gauss-Seidel correction becomes

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} + h \begin{bmatrix} -y_{n-1} \\ x_n \end{bmatrix}. \qquad (6)$$

The difference is subtle: the x_{n-1} on the bottom right gets replaced by x_n, and that's all! However since x_n is now on the right and the left we need to do a little algebra to get an explicit iteration. This results in

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -h \\ h & 1-h^2 \end{bmatrix} \cdot \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \qquad (7)$$

(see if you can work out the details!). Now there is a little bit of matrix magic. Let's call the matrix in (7) $\mathbf{A}$ and then let

$$\mathbf{B} = \begin{bmatrix} 1 & -h/2 \\ -h/2 & 1 \end{bmatrix}.$$

It can be checked (with some algebraic sweating) that there is a (strange) relationship between these two matrices, specifically that $\mathbf{A}^t\mathbf{B}\mathbf{A} = \mathbf{B}$. What does this mean for our iteration (7)? Well, then (using (7)),

$$[x_n, y_n]\mathbf{B}\begin{bmatrix} x_n \\ y_n \end{bmatrix} = [x_{n-1}, y_{n-1}]\mathbf{A}^t\mathbf{B}\mathbf{A}\begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} =$$

$$= [x_{n-1}, y_{n-1}]\mathbf{B}\begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = [x_{n-2}, y_{n-2}]\mathbf{B}\begin{bmatrix} x_{n-2} \\ y_{n-2} \end{bmatrix} = \ldots$$

$$\ldots = [x_0, y_0]\mathbf{B}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = [1,0]\mathbf{B}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.$$

In other words, the points (x_n, y_n) stay on the curve

$$1 = [x, y]\begin{bmatrix} 1 & -h/2 \\ -h/2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 - hxy + y^2$$

which happens to be an ellipse (for $h < 2$) rotated 45° counterclockwise with respect to the *x*-axis. See if you can verify this! By the way this ellipse has major axis length of
$2\sqrt{2/(2-h)}$ and minor axis length $2\sqrt{2/(2+h)}$.
For a reasonably small *h* the ellipse and the unit circle are hard to tell apart!

3. Conclusions

I hope that this little example shows that care needs to be taken when using numerical models. The underlying physics can easily be lost. Sometimes it can be recovered, as by the Gauss correction for this example, but every case needs to be carefully examined.

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Vento (*di Attilio Bertolucci - Tratto da Sirio* (1929) – *Opere di poesia*)

Come un lupo è il vento
che cala dai monti al piano,
corica nei campi il grano
ovunque passa è sgomento.
Fischia nei mattini chiari
illuminando case e orizzonti,
sconvolge l'acqua nelle fonti
caccia gli uomini ai ripari.
Poi, stanco s'addormenta e uno stupore
prende le cose, come dopo l'amore.