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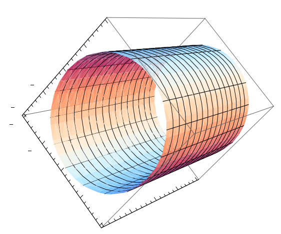

Blossoms – a Flowery Approach to Polynomials

Len Bos [*]

Take any polynomial, for example $p(x) = x^2 + 3x + 5$. This happens to be of degree $n = 2$, and there is an associated polynomial of $n = 2$ variables $P(x_1, x_2) = x_1x_2 + 3(x_1 + x_2) / 2 + 5$ called the *blossom* of $p(x)$. At first glance it seems a silly thing to do, but it has some interesting properties. First note that $P(x, x) = x{\cdot}x + 3(x + x) / 2 + 5 = p(x)$, i.e., you can recover $p(x)$ by evaluating $P(x_1, x_2)$ on the *diagonal*, $x_1 = x$, $x_2 = x$. Secondly, note that $P(x_1, x_2) = P(x_2, x_1)$ i.e., P is a *symmetric* function of its arguments. There is a third (important) property that is perhaps a bit less obvious. If you fix the first variable $x_1 = a$, then

$P(a, x_2) = ax_2 + 3(a + x_2) / 2 + 5 = (a+3/2)x_2 + (3a / 2 + 5)$,

i.e., you get a polynomial of degree *one* in the free variable x_2! Similarly, if you fix $x_2 = b$, then you get $P(x_1, b) = bx_1 + 3(x_1 + b) / 2 + 5$, which is a polynomial of degree *one* in the free variable x_1. The fact of being degree one can be characterised in a slightly more abstract (but useful!) way. If $q(x) = ax + b$ is some polynomial of degree one, we sometimes say that it is a *linear*, because it's graph is a straight line. But, we need to be careful since such a function (when $B \neq 0$) does not have the *linear mapping property*, i.e., it is *not* the case that $q(\alpha x + \beta y) = \alpha\, q(x) + \beta\, q(y)$ for all real values x, y and coefficients α, β (check it, it's easy to see!). However, a polynomial of degree one does have another (closely related) mapping property. Consider $q(\alpha x + \beta y)$ with the *restriction* that $\alpha + \beta = 1$. Then

$$q(\alpha x + \beta y) = a\,(\alpha x + \beta y) + b = a\,(\alpha x + \beta y) + (\alpha + \beta)\, b$$
$$= \alpha\,(ax + b) + \beta\,(ay + b) = \alpha\, q(x) + \beta\, q(y).$$

This (with the restriction that $\alpha + \beta = 1$) is called the *affine mapping property*. Translated to our blossom, this means that the blossom $P(x_1, x_2)$ is affine in each variable separately, i.e., *biaffine*.

In general, if $p(x)$ is a polynomial of degree n then the blossom is a polynomial of n variables $P(x_1, x_2, \ldots, x_n)$ with the following three properties:

1. Reproduction on the diagonal: $p(x) = P(x, x, \ldots, x)$.
2. Symmetry: $P(x_1, x_2, \ldots, x_n)$ is a symmetric function of its variables, i.e., you can change the order of the arguments in any possible way and still get the same result.
3. Multi-affinity: if all the variables are fixed except for one of them, say x_i, then P is a polynomial of degree *one* in x_i. In other words P has the affine mapping property in each variable separately, i.e.,
   $P(x_1, \ldots, x_{i-1}, \alpha x + \beta y, x_{i+1}, \ldots, x_n)$
   $= \alpha\, P(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) + \beta\, P(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$
   for any α, β such that $\alpha + \beta = 1$.

Before we continue, here's another example.
Take $p(x) = x^3 + 3x^2 + 4x + 5$, a cubic. The blossom is therefore of three variables and turns out to be $P(x_1, x_2, x_3) = x_1x_2x_3 + 3(x_1x_2 + x_1x_3 + x_2x_3) / 3 + 4(x_1 + x_2 + x_3) / 3 + 5$.
See if you can check that this has the three properties!

We have been talking about *the* blossom. In fact the blossom always *exists* and it is *uniquely* determined by the three properties 1., 2. and 3., but we won't go into the details here. Also, it's worth commenting that the blossom gives an alternative representation of a polynomial with some trade-offs in complication. On the one hand, the blossom has n variables, as opposed to just one! But then, with respect to each variable separately, the behaviour is that of the simplest case of a degree *one* polynomial. This has some great advantages! Here is one. The derivative of a degree one polynomial $p(x) = Ax + B$ is particularly simple – it's just the coefficient of x, i.e., $p'(x) = A$. This can be computed without using any calculus – it's just the slope of the line which can be calculated as the ratio of the *rise* to the *run* between *any* two points $x = a$ and $x = b$, i.e., $p'(x) = (p(b) - p(a)) / (b - a)$. This applies to the blossom! It turns out that, for any $a \neq b$,

$p'(x) = n\,(P(b, x, x, \ldots, x) - P(a, x, x, \ldots, x)) / (b - a)$.

Let's check this for our first example, $p(x) = x^2 + 3x + 5$. Obviously, by ordinary calculus, $p'(x) = 2x + 3$. Our formula using the blossom gives $p'(x) = 2(P(b, x) - P(a, x)) / (b - a)$ which, by the formula given above for P, becomes

$2([bx + 3(b + x) / 2 + 5] - [ax + 3(a + x) / 2 + 5]) / (b - a) = 2(b - a)\,(x + 3/2) / (b - a) = 2x + 3$.

See if you can *prove* this formula!

Now for another usage. For a point $x \in [a, b]$, we can set $t = (x - a) / (b - a)$. The value of t gives the percentage of how far x is from a relative to the distance from b to a. The value of $t = 0$ corresponds to $x = a$ and the value of $t = 1$ corresponds to $x = b$ and as t varies from 0 to 1, x varies from a to b. If you know the value of t you can recover the value of x by the simple formula $x = tb + (1 - t)a$ and so t is called the local coordinate of x with respect to the interval $[a, b]$. Note that we have $x = \alpha a + \beta b$ with $\alpha = 1 - t$ and $\beta = t$ having the property that $\alpha + \beta = 1$ and so we have written x as an affine combination (sometimes also called convex combination) of a, b. Now suppose that we want to graph a quadratic polynomial $q(x)$ for the values of x between a and b, i.e., $\{(x, q(x)) \mid a \leq x \leq b\}$. We can do this as follows. Supposing that $Q(x_1, x_2)$ is the blossom of $q(x)$, we can consider the blossom of the pair $(x, q(x))$, $P(x_1, x_2) = ((x_1 + x_2) / 2, Q(x_1, x_2))$ and then calculate, setting $x = tb + (1 - t)a$, $(x, q(x)) = P(x, x) = P(tb + (1 - t)a, tb + (1 - t)a)$.

Now, fix the second variable and do the affine expansion in the first variable to get

$t\, P(b, tb + (1 - t)a) + (1 - t)\, P(a, tb + (1 - t)a)$.

Then expand the two blossoms in the second variable to get, after a bit of algebra using the properties of blossoms,

$(x, q(x)) = t^2 P(b, b) + 2t(1 - t)P(a, b) + (1 - t)^2 P(a, a)$.

Clearly the value depends only on the *three* values $P(a, a)$, $P(a, b)$ and $P(b, b)$. Note that, since blossoms reproduce the original polynomial on the diagonal, $P(a, a) = (a, q(a))$ is the beginning point of the curve segment and $P(b, b) = (b, q(b))$ is its terminal point.

We can organize the calculation a bit better. Let $b_0 = P(a, a)$, $b_1 = P(a, b)$ and $b_2 = P(b, b)$, then (again after a bit of algebra)
$(x, q(x)) = t(tb_2 + (1 - t)b_1) + (1 - t)\,(tb_1 + (1 - t)b_0)$
and if we also set $b_0^1 = tb_1 + (1 - t)b_0$ and $b_1^1 = tb_2 + (1 - t)b_1$ then we have $(x, q(x)) = tb_1^1 + (1 - t)b_0^1$
and so we can calulate the values on the curve by first calculating b_0^1 and b_1^1 (by a simple affine combination) and then $(x, q(x))$ by the *same* affine combination! Doing this is called the Bezier approach to curves. If you want to know more (and see some pictures!), see for example, On Bezier and Subdivision Curves, MatematicaMente, Nos. 171, 172 and 173 (2012).

To finish, were you wondering what geometrical meaning the value of $P(a, b)$ had? Well, here's a little exercise: show that $P(a, b)$ is the intersection of the line tangent to the parabola at $x = a$ with the line tangent to it a $x = b$.

[*] Professore Ordinario di Analisi Numerica, Università degli Studi di Verona. E-mail: leonardpeter.bos@univr.it

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I numeri $\mathbf{P}(k, m)$

di Gabriele Pupolin [**]

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Per $k \geq 1$ e $m \geq 1$ sono tra loro legati dalla seguente relazione

$$\mathbf{P}(k + 1, m) = (k + m)\,\mathbf{P}(k, m - 1) + m\,\mathbf{P}(k, m). \qquad (4)$$

Dimostrazione.
Applicando le definizioni per $\mathbf{P}(k, m)$ e $\mathbf{P}(k, m - 1)$

$$\mathbf{P}(k,m) = \sum_{i=0}^{m} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\} = \sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\};$$

$$\mathbf{P}(k,m-1) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1+k}{i} \left\{\begin{matrix} m-1+k-i \\ m-1-i \end{matrix}\right\} = \sum_{i=0}^{m-2} (-1)^i \binom{m-1+k}{i} \left\{\begin{matrix} m-1+k-i \\ m-1-i \end{matrix}\right\},$$

si applica la (4) per ricavare il valore di $\mathbf{P}(k + 1, m)$.

$$m\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\} + (m+k)\sum_{i=0}^{m-2} (-1)^i \binom{m-1+k}{i} \left\{\begin{matrix} m-1+k-i \\ m-1-i \end{matrix}\right\}.$$

Ponendo nella seconda sommatoria $(i + 1) = i'$ si ottiene:

$$= m\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} (m+i-i) \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\} + -(m+k)\sum_{i'=1}^{m-1} (-1)^{i'} \binom{m-1+k}{i'-1} (i+1) \left\{\begin{matrix} m+k-i' \\ m-i' \end{matrix}\right\};$$

riportando $i' = 1$ nella seconda sommatoria e ricordando che

$$(m+k)\binom{m-1+k}{i-1} = i\binom{m+k}{i}$$

si ottiene

$$= m\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} (m-i) \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\} + -\sum_{i=1}^{m-1} (-1)^i\, i \binom{m+k}{i} \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\}.$$

Estendendo la seconda sommatoria da $i = 0$ e raccogliendo i termini comuni nelle due sommatorie si ottiene:

$$\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} (m-i) \left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\}.$$

Ricordando che

$$(m-i)\left\{\begin{matrix} m+k-i \\ m-i \end{matrix}\right\} = \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} - \left\{\begin{matrix} m+k-i \\ m-1-i \end{matrix}\right\}$$

si ottiene

$$\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} + -\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i \\ m-1-i \end{matrix}\right\}.$$

Nella seconda sommatoria si può reindicizzare $i = i' - 1$ ottenendo

$$\sum_{i=0}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} + +\sum_{i'=1}^{m} (-1)^{i'} \binom{m+k}{i'-1} \left\{\begin{matrix} m+k-i'+1 \\ m-i' \end{matrix}\right\}.$$

Se si riporta $i' = i$ nella seconda sommatoria e la si estende sino a $(m - 1)$, isolando nella prima sommatoria il termine $i = 0$, si ottiene:

$$\sum_{i=1}^{m-1} (-1)^i \binom{m+k}{i} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} + +\sum_{i=1}^{m-1} (-1)^i \binom{m+k}{i-1} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} + \left\{\begin{matrix} m+k+1 \\ m \end{matrix}\right\} =$$

$$= \sum_{i=1}^{m-1} (-1)^i \binom{m+k+1}{i} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\} + \left\{\begin{matrix} m+k+1 \\ m \end{matrix}\right\} =$$

$$= \sum_{i=0}^{m-1} (-1)^i \binom{m+k+1}{i} \left\{\begin{matrix} m+k-i+1 \\ m-i \end{matrix}\right\}.$$

La sommatoria

$$\sum_{i=0}^{m-1} (-1)^i \binom{m+k+1}{i} \left\{\begin{matrix} m+k+1-i \\ m-i \end{matrix}\right\} =$$

$$\sum_{i=0}^{m} (-1)^i \binom{m+k+1}{i} \left\{\begin{matrix} m+k+1-i \\ m-i \end{matrix}\right\} = \mathbf{P}(k+1, m)$$

in quanto per $i = m$ si ha

$$\left\{\begin{matrix} m+k+1-i \\ m-i \end{matrix}\right\} = 0.$$

Resta così dimostrata la (4).
Si calcolano i valori dei $\mathbf{P}(k, m)$ per i valori di: $k = 0$, $m = 0$, $k = m$ e $m = 1$.

Per $\mathbf{P}(0, 0)$ si ha:

$$\mathbf{P}(0,0) = \sum_{i=0}^{0} (-1)^i \binom{0}{i} \left\{\begin{matrix} 0-i \\ 0-i \end{matrix}\right\} = \mathbf{1}\cdot\mathbf{1} = \mathbf{1}; \qquad (5)$$

per $\mathbf{P}(0, m)$ con $m > 0$, si ha:

$$\mathbf{P}(0,m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\{\begin{matrix} m-i \\ m-i \end{matrix}\right\} =$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} = \mathbf{0} \qquad (6)$$

per $\mathbf{P}(k, 0)$, con $k > 0$, si ha:

$$\mathbf{P}(k,0) = \sum_{i=0}^{0} (-1)^i \binom{k}{i} \left\{\begin{matrix} k-i \\ 0-i \end{matrix}\right\} = \mathbf{1}\cdot\mathbf{0} = \mathbf{0}; \qquad (7)$$

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[**] Socio nazionale Mathesis - Preside CIFI (Collegio Ingegneri Ferroviari Italiani) – Venezia – e-mail: gabriele.pupolin@gmail.com