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Modelling the Future

Len Bos [*]

It is said that only fools and charlatans (and climatologists) try to predict the future. What about Mathematicians? Well, they instead try to *model* the *variability* of the future. This may be regarded as only one step removed from foolishness, but it does turn out to be a useful idea. Here is an example. Suppose that the current temperature is 0°C and we're pretty sure that within the next hour the temperature can change by at most 1 degree, up or down, i.e., go up to at most +1°C or down to at most –1°C.

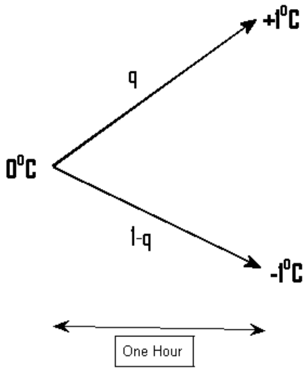

Figure One – A Single Time Step

Now, we are not going to say that the temperature *will* go up or down. But we are going to say that it *might* go up or down and we try to estimate the likelihood of which by assigning a probability q for an up tick of one degree and hence probability $1-q$ for a down tick of one degree. This is our simplest possible model – only two possible outcomes, each with a certain probabilty and is rather limited. For example, there is no room in our model for the temperature *not* to change, it has to either go up or down by one degree – there are no other possibilities. However, we can already compute the expected temperature, i.e., the average of the temperature in one hour:

$$E(T_1) = q(+1) + (1-q)(-1) = 2q - 1$$

and its variance:

$$Var(T_1) = E((T_1 - E(T_1))^2) = 4q(1-q)$$

(after a little calculation). This at least gives us something concrete to work with. For example, the variance measures the level of uncertainity and we can see that it is 0 when $q = 0$ (i.e., we are absolutely positive that the temperature will drop) and when $q = 1$ (i.e., we are absolutely positive that the temperature will go up). The uncertainity is greatest when the variance is at a maximum which occurs exactly when $q = 1/2$ (can you verify this?). This makes perfect sense as this is the case when we can't give any preference to whether the temperature goes up or down. But still such a model is not very realistic. How can we make a better one?

A First Attempt

One way is to consider the possibility of intermediate temperature changes, say every half-hour instead of every hour. Now, we have to be a bit careful. We can't allow a change of *one* degree every half-hour as that would allow for a possible change of two degrees after an hour. Hence if we double our number of time steps (1 hour to ½ hour) we should divide our temperature jumps by a half to $1/2$ degree per half-hour, or so it would seem! Figure two shows what is happening (we still let q denote the probability of an increase). There are now *three* possible outcomes for the temperature in one hour: $+1$, 0 and -1. Notice that we are also modelling the intermediate temperature and we have *four* possible *scenarios*:

$$\{+,+\}, \quad \{+,-\}, \quad \{-,+\}, \quad \{-,-\}.$$

To get to $+1$ the temperature would have to increase both times, each with probability q and so the probability of $T_1 = +1$ is $q \times q = q^2$.

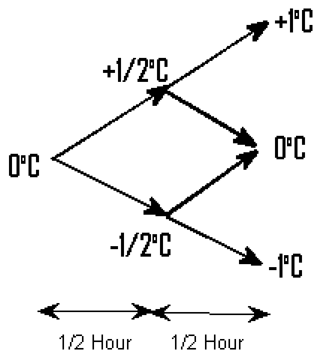

Figure Two – Two Time Steps

There are two different scenarios that lead to temperature 0: the scenario $\{+,-\}$, i.e., go up by $1/2$ degree in the first half hour and then down by $1/2$ degree in the second half-hour (with probability $q(1-q)$) and the scenario $\{-,+\}$, i.e., go down by $1/2$ degree in the first half-hour and then up by $1/2$ degree in the second half-hour (with a probability of $(1-q)q$). Hence the probability of $T_1 = 0$ is $2q(1-q)$. Lastly, to end up at -1 the temperature would have to drop during both half-hours (scenario $\{-,-\}$) and so the probability of $T_1 = -1$ is $(1-q)^2$. This is already a slightly more sophisticated model, with three possible values for the temperature in one hour. You can check that the sum of the three probabilities

$$q^2 + 2q(1-q) + (1-q)^2 = 1$$

which is equivalent to saying that one of these three possibilities will surely occur. Still not highly realistic, but at least better than just two outcomes! Again, we can calculate the mean

$$E(T_1) = q^2(+1) + 2q(1-q)(0) + (1-q)^2(-1) = 2q - 1,$$

the same as before(!) and the variance

$$Var(T_1) = 2q(1-q).$$

But now the variance is smaller! Hmm. Does this make sense? Or is there some mistake? Let's look at the general case and try to figure out what's going on. Consider dividing our hour into n equal times of length $1/n$ during which the temperature could go up by $+1/n$ degrees with probability q and down by $-1/n$ degrees with probability $1 - q$. We then have 2^n different scenarios of n ups and downs, e.g.,

$$\{+,+,-,-,-,+,\ldots,-\}.$$

For each scenario the final temperature in one hour is the sum of all the intermediate changes. Hence if there are j "+"s and $n-j$ "−"s in the scenario then

$$T_1 = j\left(\frac{1}{n}\right)+(n-j)\left(-\frac{1}{n}\right)=\frac{2j}{n}-1$$

In particular T_1 depends just on j, the number of "+"s, and *not* on the order in which they occurred! Further, since there are

$$\binom{n}{j}$$

ways to select exactly j "+"s in a string of n "+"s and "−"s, and the probability of this is $q^j(1-q)^{n-j}$, the probaility of

$$T_1=\frac{2j}{n}-1 \quad \text{is} \quad \binom{n}{j}q^j(1-q)^{n-j}$$

and we have what is called a Binomial Distribution for the temperature in one hour. We can calculate the mean

$$E(T_1)=\sum_{j=0}^{n}\left(\frac{2j}{n}-1\right)\binom{n}{j}q^j(1-q)^{n-j}$$

$$=2\sum_{j=0}^{n}\left(\frac{j}{n}\right)\binom{n}{j}q^j(1-q)^{n-j}-1=2q-1\,,$$

just as for n = 1. So far so good. But the problem was with the variance, so we calculate

$$Var(T_1)=\sum_{j=0}^{n}\left(\frac{2j}{n}-1\right)^2\binom{n}{j}q^j(1-q)^{n-j}-(2q-1)^2$$

$$=\frac{4q(1-q)}{n}\,,$$

after some algebra. (Can you verify this?) Now this is revealing! As n increases the variance of T_1 goes to zero! This means that the distribution starts concentrating more and more about the mean, and all other possibilities become less and less likely. Is this really what we want in a model? I doubt it! It doesn't appear to at all be realistic. This is something that we should always be on the lookout for when modelling something. What appear to be reasonable assumptions can result in unexpected consequences! Our model is wrong! So what do we do? [*To be continued in number* 204]

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… Terribilis est locus iste …

di Luciano Corso

[*Segue dal numero* 200]

È una donna e mi sorride, ma è un sogno. Nei sogni possono capitare molte cose; innamorarsi, per esempio, fare errori su errori, diventare padri, fare affari e guadagnarci pure, appassionarsi, perdere tutto al gioco.

Anche nella vita può accadere tutto ciò. La vita è un sogno? Mah! Vorrei svegliarmi da questo sogno. Cinque punti son pochi. Com'è possibile passare il tempo fino al risveglio? Quanti possibili percorsi distinti posso fare con la mia navicella con vertici questi 5 punti partendo da un vertice per arrivare infine allo stesso? La risposta è data da tutte le iniezioni da un 5-insieme a un 5-insieme: 5! = 120 percorsi distinti.

Consideriamo ora le permutazioni di n elementi e definiamo il concetto di ciclo. Per ciclo intendo un movimento che nella permutazione considerata, a partire da un elemento ritorna nello stesso, dopo un certo numero di passi.

Quanti cicli ci sono in ciascuna permutazione? C'è un sistema ricorsivo che conta il numero di permutazioni distinte $s(n, k)$ di k cicli che si possono ottenere a partire da un insieme lungo n:

$$\begin{cases} s(n,0)=0 \\ s(n,n)=s(0,0)=1 \\ s(n+1,k)=s(n,k-1)+n\,s(n,k) \end{cases} \quad (1)$$

dove $n > 0$, $0 < k \leq n$ (Si giustifichi, dal punto di vista combinatorio la 3ª riga di (1)). Per valutare quante permutazioni distinte di k cicli si possono fare con un insieme di 5 punti {a, b, c, d, e} si procede così:

s(0,0) = 1, s(1,0) = 0, s(1,1) = 1,
s(2,0) = 0, s(2,2) = 1,
s(2,1) = s(1,0) + 1 s(1,1) = 1,
s(3,0) = 0,
s(3,1) = s(2,0)+2·s(2,1) = 0 + 2·1 = 2
s(3,2) = s(2,1)+2·s(2,2)=1+2·1 = 3,
s(3,3) = 1,
s(4,0) = 0,
s(4,1) = s(3,0) + 3·s(3,1) = 0 + 3·2 = 6
s(4,2) = s(3,1) + 3·s(3,2) = 2 + 3·3 = 11
s(4,3) = s(3,2) + 3·s(3,3) = 3 + 3·1 = 6
s(4,4) = 1,
s(5,0) = 0
s(5,1) = s(4,0) + 4·s(4,1) = 0 + 4·6 = 24
s(5,2) = s(4,1) + 4·s(4,2) = 6 + 4·11 = 50
s(5,3) = s(4,2) + 4·s(4,3) = 11 + 4·6 = 35
s(5,4) = s(4,3) + 4·s(4,4) = 6 + 4·1 = 10
s(5,5) = 1.

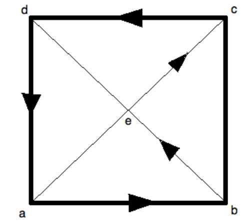

Fig. 1. Lo schema descrive 5 dei 24 possibili percorsi che posso fare partendo da un punto e ritornando là: (a, b, e, c, d, a), (b, e, c, d, a, b), (c, d, a, b, e, c), (d, a, b, e, c, d), (e, c, d, a, b, e), toccando tutti gli elementi dell'n-insieme; siamo in presenza di permutazioni a ciclo uno.

Scrivo la tabellina riepilogativa su di un foglio presente nel contenitore cilindrico:

$s(n, k)$		n					
		0	1	2	3	4	5
k	0	1	0	0	0	0	0
	1		1	1	2	6	24
	2			1	3	11	50
	3				1	6	35
	4					1	10
	5						1

La (1) da dove proviene? Tutto parte dal concetto di iniezione, in questo caso, da un n-insieme (particelle?) a un $(x + n - 1)$-insieme (celle?). Le iniezioni che si ottengono sono:

$$x^{\lfloor n \rfloor}=x(x+1)(x+2)\ldots(x+n-1)\,. \quad (2)$$

Oltre al significato combinatorio, (2) è un polinomio di grado n. Se lo sviluppo, ottengo in generale:

$$x^{\lfloor n \rfloor}=\sum_{k=0}^{n}a(n,k)x^k \quad (3)$$

dove $a(n, k)$ sono i coefficienti del polinomio e l'estensione a $k = 0$ non muta alcunché. Tenendo conto di (3) e che

$$x^{\lfloor n+1 \rfloor}=(x+n)x^{\lfloor n \rfloor} \quad (4)$$

formo l'identità

$$\ldots+a(n+1,k)x^k+\ldots=\ (x+n)\left(\ldots+a(n,k-1)x^{k-1}+a(n,k)x^k+\ldots\right)$$

da cui:

$$a(n+1,k)=a(n,k-1)+n\,a(n,k)\ . \quad (5)$$

I coefficienti $a(n, k)$ di (3) soddisfano (1), quindi coincidono con $s(n, k)$; attribuiamo pertanto a questi ultimi un significato algebrico oltre che un significato combinatorio.

[*Segue al numero* 208]