ISSN: 2037-6367

MatematicaMente

Pubblicazione mensile della sezione veronese della MATHESIS – Società Italiana di Scienze Matematiche e Fisiche – Fondata nel 1895 – Autorizzazione del Tribunale di Verona n. 1360 del 15 – 03 – 1999 – I diritti d'autore sono riservati. Direttore: Luciano Corso – Redazione: Alberto Burato, Fabrizio Giugni, Michele Picotti, Sisto Baldo – Via IV Novembre, 11/b – 37126 Verona – tel e fax (045) 8344785 – 338 6416432 – e-mail: lcorso@iol.it – info@mathesisverona.it – Stampa in proprio – Numero 206 – Pubblicato il 03 – 11 – 2015

Modelling the Future and Financial Option Prices

Len Bos [*]

In our previous article **Modelling the Future (MTF)**, we gave a simple probabilistic *tree* that can be used to model the variability of future events. *Here* we will discuss a similar model, also very simple, that can be used to give "fair" prices to what might seem to be "complicated" financial instruments.

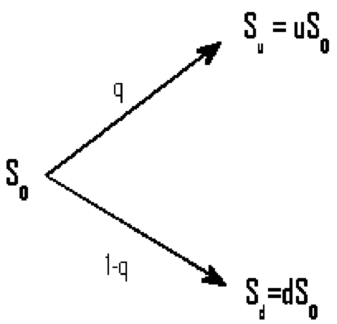

Figure One – A Single Time Step Model of the Asset

But first we should describe a bit what these financial instruments are. Specifically we will be talking about what are called (European style) Options. An Option is a contract (signed by two parties) that gives the *buyer* the right, but *not* the obligation to buy or sell a certain so-called asset (called the underlying asset) at a fixed future date and at a fixed price, both stipulated in the contract. If the Option is to *buy* then it is called a Call Option. On the other hand, if the option is to *sell* then it is called a Put Option. It is important to remark that the buyer has the choice – he/she does not have to buy/sell if he/she doesn't feel like it! The underlying asset could in principle be anything that can be bought and sold. Often they are company stocks, but could be other commodities such as oil, gas or even electricity!

Now, is there any value to having such an Option? Let's look at a little example. Suppose that you have an option to buy a certain asset, exactly one month from today, for a price of 100 euros. The current price is 90 euros and you suppose that the price could go up to 110 euros, unless something bad happens and it goes down to 80. What does this mean to you? First of all, *exactly* what will happen is unknown to you – the price could go up or it could go down. But if it did go up you would be able to buy an asset that was now worth 110 for the low price of only 100 euros, for a profit of 10 euros. On the other hand, if the price went down to 80 then it would make no sense to *exercise* your option – why pay 100 for something worth only 80 if you don't have to! Hence, the value to you of having the option is zero. In summary, the option in one month's time (the so-called *expiry* date) is worth either 10 euros or 0 euros, depending on what happens to the price of the underlying asset. It's like having a lottery ticket. If you win you get rich, and if you lose, well then you have to try again (or, better, give up on lotteries!). So overall, having the option is worth *something*. What we want to do is to determine how much.

First we make a model of what can happen to the underlying asset. It's very similar to the model of **Modelling the Future** with one major difference: in financial matters things tend to go up or down by percentages and so we use a *multiplicative* model (as opposed to the *additive* model of **MTF**). The model is described in Figure 1. We call S_0 the current price of the asset which could go up to an "up value" $S_u = u \times S_0$, with "up factor" u or down to a "down value" $S_d = d \times S_0$ with "down factor" d. Of course we assume that $u > d$ (so that an "up" is really better than a "down"). We use q to denote the probability of an "up" so that $1 - q$ is the probability of a "down". Now we make a corresponding model of the option, shown in Figure 2. Here $V_u = payoff(S_u)$ is the value of the option (to you) if the underlying asset goes "up", $V_d = payoff(S_d)$ is the value of the option if the asset goes "down" and V_0 is the current price of the option and *is what we are trying to find*! Note that the *payoff* is something that can be computed, depending on the details of the option, just as we did for the little example above.

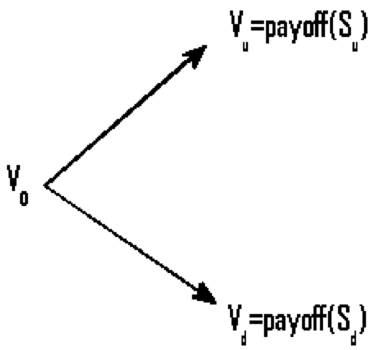

Figure Two – A Model of the Option

Now to find a "fair" value for V_0. To do this we create an artificial (imaginary) portfolio (just a collection of some financial assets) consisting of one option and $-\Delta$ underlying assets:

$$P = V - \Delta S.$$

Its value in the "up" state is $P_u = V_u - \Delta S_u$ while its value in the down state is $P_d = V_d - \Delta S_d$. But now there's something interesting that we can do: we can choose Δ so that $P_u = P_d$, i.e., so that the value of the portfolio does *not* depend on what happens to the asset, i.e., it's value is *deterministic* instead of *stochastic*!
How do we do this? It's just a little algebra: set $P_u = P_d$ and solve for Δ. If you do this you get a value of

$$\Delta = \frac{V_u - V_d}{S_u - S_d}$$

It's important to understand what this choice of portfolio gives us! Another way of saying that its value is deterministic is to say that it's value is *risk free*, we know (within the confines of this model) *for sure* what its value will be at the expiry date. Now there comes into play an Economic/Financial Principle, the so-called Law of One Price. This states that any two riskless assets must have the same rate of return (i.e. the same profit percentage). If this were not true it would be possible to make money without taking any kind of risk (so-called arbitrage). Here is a simple example. Suppose you have bank accounts in two different banks. The first one is willing to give you 10% interest on your investment, while the second one is willing to lend you money at the rate 5%. What would you do? Well, you could borrow money from the second bank and invest it in the first one. It will cost you 5% to borrow, but you get a return of 10% leaving you with a clean profit of 5%, without at any time having risked anything!

Ok, let's assume this Law of One Price. With our choice of Δ we have a riskless portfolio P. What other riskless investments are there? There are the banks! Now, technically money in the bank is not 100% riskless, but as idealization it's about as riskless as you are ever going to get (notwithstanding the current situation in Greece!). Let's say that the return on a bank deposit over the same period is R, i.e., if we invest 1 euro we will get $1 \times R$ euros at the expiry time. Then the Law of One Price says that our riskless portfolio P should have the same return, i.e., at expiry its value should be

$$R \times P_0.$$

But at expiry, with our special choice of Δ, the value of the portfolio is $P_u = P_d$ (either one, as they are equal). Hence

$$R \times (V_0 - \Delta S_0) = R \times P_0 = P_u = V_u - \Delta S_u$$

which we can solve for V_0, the quantity we are interested in:

$$V_0 = \Delta S_0 + R^{-1}(V_u - \Delta S_u)$$

where Δ is the special value given above.

What then is the value of the option in our little example? Suppose that $R = 1.1$ (i.e., we have a 10% riskless rate of return). Then we calculate $\Delta = \frac{10-0}{110-80} = \frac{1}{3}$ so that

$$V_0 = \frac{90}{3} + \frac{1}{1.1}\left(10 - \frac{1}{3}110\right) = 5.76 \text{ euros.}$$

There is an interesting way of re-writing the formula for V_0. Indeed, what we do is see how it depends on the two values V_u and V_d. After a bit of algebra we obtain

$$V_0 = R^{-1}(pV_u + (1-p)V_d)$$

Where

$$p = \frac{R-d}{u-d}.$$

Some interpretation is in order. The factor R^{-1} is a so-called discounting factor – it gives the *present value* of money, having X euros at the expiry date is the same, financially, as having $R^{-1} \times X$ euros right now, since you could invest your $R^{-1}X$ euros with return R to get $R(R^{-1}X) = X$ euros at the expiry date. More importantly, we claim that p is actually a *probability*, i.e., $0 \le p \le 1$. Now why is this? First of all let's see why it cannot be the case that $p < 0$. This is the case for financial/economic reasons (and not purely mathematical ones!). Indeed, if $p < 0$ then from the formula for p, we would have $R < d$. This means that investing in the bank is worse than the worst that can happen to the asset. Hence any reasonable investor would take out all their money from the bank and invest it in this asset (it could still go "up" or "down" depending on the probability q but whatever happens its return is better than that of the bank!!). The banks not being willing to lose all their depositors will then of course raise their interest rates (or else go out of business!) so the situation will not last and is not economically tenable. Similarly, if $p > 1$, then from the fomula for p, $R - d > u - d$ so that $R > u$. What does this mean economically? Well, then investing in the bank is *better* than the best that the asset can do. Why then invest in this risky asset (it could even go down) if you can get a better return without taking a risk? The asset will disappear from the market!

This probability p is called the *risk-neutral probability* and helps to organize your thinking about option pricing much like Newton's Laws helps to organize your thinking about classical Mechanics. Indeed, the expression $pV_u + (1-p)V_d$ is the *mean* of V_u and V_d with respect to the probabilty p and hence we may write

$$V_0 = R^{-1}\mathbb{E}_p(V_{future}) \qquad (1)$$

i.e, *the value of an option is the discounted mean of its future value with respect to the risk-neutral probability*.

Now why use this strange name, the risk-neutral probability? There's a good reason! First of all, it's an artificial devise, not related to the *actual* probability q (sometimes called the *real-world* probability), and introduced, again, for the purpose of organizing our thoughts (something like introducing the concept of mass in Mechanics). Secondly, p has some peculiar properties with regard to risk. Now, we actually are dealing with three different financial instruments: 1. The underlying asset, 2. Money in the bank and 3. Our Option on the underlying asset. Each of these has its own level of risk (with the bank having zero risk). We have the formula (1) for the value of the Option in terms of p. Let's see what happens when we apply the same formula to the other two assets. First, for the underlying

$$\begin{aligned} R^{-1}\mathbb{E}_p(S_{future}) &= R^{-1}(pS_u + (1-p)S_d) \\ &= R^{-1}\left(\frac{R-d}{u-d}\, S_u + \frac{u-R}{u-d} S_d\right) \\ &= R^{-1}\left(\frac{R-d}{u-d}\, (uS_0) + \frac{u-R}{u-d} (dS_0)\right) \\ &= R^{-1}S_0 \frac{(R-d)u + (u-R)d}{u-d} \\ &= R^{-1}S_0 \frac{R(u-d)}{u-d} = S_0, \end{aligned}$$

i.e., it is also the case that the current price of the asset is the discounted of the mean of its future value under the probability p! Secondly, for money in the bank

$$R^{-1}\mathbb{E}_p(M_{future}) = R^{-1}(pM_u + (1-p)M_d)$$

where M_u denotes the value of an investment in the "up" state of the asset and M_d in the "down" state. But for money in the bank, $M_u = M_d = RM_0$ so that

$$R^{-1}\mathbb{E}_p(M_{future}) = R^{-1}RM_0(p + (1-p)) = M_0$$

and we have that also the current value of money in the bank is the discounted mean of its future value with respect to the probability p! In other words, the same formula, using the special, artificial probability p, holds for all three of our assets, each having its own level of risk. In this sense p is blind to the level of risk!

The above method of using the portfolio $P = V - \Delta S$ and choosing Δ to eliminate the risk is called the *method of the riskless portfolio* and the process of choosing Δ is called *hedging*. But there is another way that is very useful for practitioners, that is, the method of the *duplicating portfolio*. It works like this. We now make a portfolio $P = aS + b$ consisting of a certain number a of the underlying asset and a certain number b of euros in the bank. The idea is to choose a, b so that the value of the portfolio mimics the value of the option. Now at expiry there are two possibilities, either the asset goes "up" or it goes "down", for which there are two possible values for the option V_u and V_d and two possible values for the portfolio P_u and P_d. Setting these corresponding values equal to each other gives us two (linear) equations in the two unknowns a, b from which they should be determined. Here are the details. We have

$$\begin{aligned} &P_u = V_u \text{ and } P_d = V_d \\ \Leftrightarrow\ &aS_u + Rb = V_u \text{ and } aS_d + Rb = V_d \\ \Leftrightarrow\ &a = \frac{V_u - V_d}{S_u - S_d} = \Delta \text{ and } b = R^{-1}(V_u - aS_u). \end{aligned}$$

Now with this choice of a, b, again by the Law of One Price (see if you can fill in the details!) we must also have

$$V_0 = P_0 = aS_0 + b.$$

It just takes a little algebra to see that this is the same value as before. Many of the ideas of Option Pricing are already present in the simple models we have discussed. You may well ask about what happens if we have more than one time step. That's an interesting question! Stay tuned for a future article on that subject!

[*] Professore Ordinario di Analisi Numerica, Università degli Studi di Verona. E-mail: leonardpeter.bos@univr.it