ISSN: 2037-6367

MatematicaMente

Pubblicazione mensile della sezione veronese della MATHESIS – Società Italiana di Scienze Matematiche e Fisiche – Fondata nel 1895 – Autorizzazione del Tribunale di Verona n. 1360 del 15 – 03 – 1999 – I diritti d'autore sono riservati. Direttore: Luciano Corso – Redazione: Alberto Burato, Fabrizio Giugni, Michele Picotti, Sisto Baldo, Andrea Sellaroli, Bruno Stecca – Via IV Novembre, 11/b – 37126 Verona – tel. 338 6416432 – e-mail: lcorso@iol.it – info@mathesisverona.it – Stampa in proprio – Numero 243 – Pubblicato il 05 - 12 - 2018

THE PROBLEM OF BANACH AND EQUIVALENT MODELS

by Luciano Corso [*]

Somebody has two matchboxes in his trouser pockets: A and B. Each box at the beginning has n matches inside. He randomly extracts one of the two boxes from his pockets and takes a match to light a cigarette. This process goes on until one of the two boxes is empty. One wonders what is the probability that after having exhausted the matches of one of the two boxes, still a number r of them remains in the other. The quantity of matches that he extracts is equal to 2n – r and the remaining r are all in the same box (either in A or in B).

Negative binomial

A mathematical model able to describe and solve the Banach problem starts from the following considerations. The boxes *A* and *B* are arranged in two distinct pockets. One of the two boxes is randomly drawn and from this one is then extracted a match. The process continues in this way until the first empty box is extracted (we assume that after the last match extracted the box is replaced in the pocket without particular attention). One wonders what is the probability that at the extraction of the empty box, the other has *r* matches.

The random variable that describes the phenomenon is the negative binomial: we leave to the reader the task of learning more about it by consulting the bibliographical references [B.1], [B.2], [B.3], [B.4].

Let's try to follow a somewhat different and more analytical reasoning. On closer examination, *A* and *B* represent the letters of a binary alphabet with which we form a word of length $2n - r + 1$. Furthermore, there are two possible constructions of this word: the one presenting $(n + 1)$ times the event A and $(n - r)$ the event B and vice versa.

Let's present both of them.

The first word has $|A| = n + 1$ and $|B| = n - r$:

$$A_1\, A_2\, A_3 \dots A_n\, B_1\, B_2 \dots B_{n-r} \mid A_{n+1} \qquad (1)$$

where the subscripts of (1) count how many times the events A and B have happened; in particular, if $r = 0$ then the events A and B have happened *n* times each and if $r = n$ then B_0 means that B has never occured.

The second word has $|A| = n - r$ and $|B| = n + 1$:

$$B_1\, B_2\, B_3 \dots B_n\, A_1\, A_2 \dots A_{n-r} \mid B_{n+1}. \qquad (2)$$

where the subscripts of (2) are interpreted as above.

Now, while in the two words the first $2n - r$ events are interchangeable in all possible ways (the order does not count), the last event, respectively *A* or *B* to the test $(2n - r + 1)$, is bound (note the logical condition symbol "|" in (1) and (2)). It is in fact the case in which one picks up an empty box. Both groups of two words lead to the same result. In the first and also in the second case we can form the following quantity of words:

$$\binom{2n-r}{n-r},$$

with A_{n+1} or B_{n+1} constrained to exit the last test: the $(2n - r + 1)$-th. The probabilities that *A* or *B* occur in each test are constant and therefore the probability sought is given by

$$\text{Prob}[(A_1, \dots, A_n, B_1, \dots, B_{n-r} \mid A_{n+1}) \cup (B_1, \dots, B_n, A_1, \dots, A_{n-r} \mid B_{n+1})] \qquad (3)$$

and so we obtain:

$$\frac{1}{2}\binom{2n-r}{n-r}\frac{1}{2^{2n-r}} + \frac{1}{2}\binom{2n-r}{n-r}\frac{1}{2^{2n-r}} = \binom{2n-r}{n-r}\frac{1}{2^{2n-r}}\ . \qquad (4)$$

With $n = 4$ matches and $r = 2$, we have:

$$\binom{6}{2}2^{-6} = \frac{15}{64}\ .$$

The phrase "having found that a box is empty" has an ambiguity: do you make this finding by extracting the last match from one of the two boxes or when extracting an empty box? We have hypothesized not to notice having arrived at an empty box at the extraction of the last match. If, however, the process stops at the extraction of the last match of a box, then reasoning by analogy starting from (3) and (4) one would have:

$$\text{Prob}[(A_1, \dots, A_{n-1}, B_1, \dots, B_{n-r} \mid A_n) \cup (B_1, \dots, B_{n-1}, A_1, \dots, A_{n-r} \mid B_n)] \qquad (5)$$

and then:

$$\binom{2n-r-1}{n-r}\frac{1}{2^{2n-r-1}}\ . \qquad (6)$$

Applying the relation to our example we have:

$$\binom{5}{2}2^{-5} = \frac{20}{64}\ .$$

Emptying an energy system in *n* steps

The most bizarre models of probability calculation are often used as good approximations of certain topics in Physics, Biology, Economics. Here we present a thermodynamic approach to the Banach scheme.

A thermodynamic system consists of two cells A and B in a space $\mathbf{R}^3$ such that in each cell there are *n* particles «*on*».

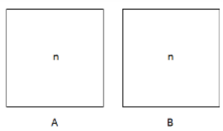

Figura 1. There are *n* particles «on» in each of the two parts *A* and *B* of the space. The number of the «on» particles defines the energy state of the two parts. There is a part external to the two parts *A* and *B*. The two parts are both covered by a permeable membrane that allows the particles "on" to pass from one part to the external environment. From each part *A* or *B* only one particle can emerge at a time in sequences of time intervals Δt_i. As time changes, the probability of obtaining one of the two parts without particles and the other with particles is given by (6). The scheme is equivalent to that of Banach.

Two membranes that surround the 2 systems allow the unidirectional passage (irreversibility of time) of the particles «on» from one of the two systems to the external environment in each time span.

The transition to the exterior occurs at random from one of the two systems at a time. The phenomenon continues until one of the two states has no more "on" particles within it (absolute vacuum of energy). What is the probability that, when this happens, in the other system the "on" particles are still r? The answer is determined by applying the prerequisite of the analogy to the two systems.

A variant

Let's face a first problem: if I come to know in some way that there are r matches still to be extracted, what is the probability that they are all placed in one of the two boxes?
The possible ways to extract $2n - r$ matches from the $2n$ that are in the two boxes are:

$$\binom{2n}{2n-r} = \binom{2n}{r}.$$

In fact, the order does not matter and there is no reposition. For example, if $n = 4$ and $r = 2$ we have:

$$\binom{2\cdot 4}{2\cdot 4-2} = \binom{8}{6} = 28\,.$$

The favorable cases, however, are:

$$\binom{n}{n}\binom{n}{n-r} + \binom{n}{n-r}\binom{n}{n} = 2\cdot\binom{n}{n-r}.$$

Table I			
A	B	A	B
0000	0011(1)	0100	0001
0000	0101(2)	0100	0010
0000	0110(3)	0100	0100
0000	1001(4)	0100	1000
0000	1010(5)	1000	0001
0000	1100(6)	1000	0010
0001	0001	1000	0100
0001	0010	1000	1000
0001	0100	0011	0000(7)
0001	1000	0101	0000(8)
0010	0001	0110	0000(9)
0010	0010	1001	0000(10)
0010	0100	1010	0000(11)
0010	1000	1100	0000(12)

Table I shows the empirical verification of the validity of the relation (7) found for $n = 4$ and $r = 2$; presents the possible events ("1" is "there is a match" and "0" is "there is not a match") and the 12 favorable events are identified with a number between brackets.
The probability sought for is:

$$\mathrm{Prob}(E|n,r) = 2\binom{n}{n-r}\Big/\binom{2n}{2n-r}, \qquad (7)$$

where E is the event that "since there are r matches in the two boxes, after $2n - r$ extractions, these are all in one of the two boxes".
In our case we have:

$$\mathrm{Prob}(E|n,r) = 2\binom{4}{4-2}\Big/\binom{2\cdot 4}{2\cdot 4-2} = \frac{12}{28}.$$

If the boxes had n_A and n_B matches, respectively, with $n_A \neq n_B$, then the possible cases would be

$$\binom{n_A+n_B}{n_A+n_B-r} = \binom{n_A+n_B}{r}$$

and the favorable cases would be:

$$\binom{n_A}{n_A}\cdot\binom{n_B}{n_B-r} + \binom{n_B}{n_B}\cdot\binom{n_A}{n_A-r} = \binom{n_B}{n_B-r} + \binom{n_A}{n_A-r}.$$

Therefore:

$$\mathrm{Prob}(E|n_A,n_B,r) = \frac{\binom{n_A}{n_A-r}+\binom{n_B}{n_B-r}}{\binom{n_A+n_B}{n_A+n_B-r}}. \qquad (8)$$

For example, in the hypothesis that in sectors A and B there are respectively $n_A = 4$ and $n_B = 2$ and that $r = 2$, then we have:

$$\mathrm{Prob}(E|n_A = 4, n_B = 2, r = 2) = \frac{\binom{4}{2}+\binom{2}{0}}{\binom{6}{4}} = \frac{7}{15}.$$

Note: The problem of the "Banach's matches" was proposed by Hugo Steinhaus (Polish mathematician) who seems to have been inspired by the habits of the mathematician Stefan Banach (1892 - 1945), also Polish, known for his studies on normed vector spaces, and a stubborn smoker, who had such a frenzy of cigarette consumption to be haunted by the fear of remaining without cigarettes and without matches for. The example soon became a classic probability problem in international publications.

References: [B.1] Letta G., *Probabilità elementare*, edizioni Zanichelli, Bologna, 1993. [B.2] Landenna G., Marasini D., Ferrari P., *Probabilità e variabili casuali*, ed. Il Mulino, Bologna, 1997. [B.3] Gnedenko B. V., *Teoria della probabilità*, Editori Riuniti, Mosca – Roma, 1979. [B.4] Feller W., *An Introduction to Probability Theory and Its Applications*, Vol. I, J. Wiley & Sons Inc., New York, 1957. [B.5] Atkins Peter W., *Il secondo principio*, ed. Zanichelli, Bologna, 1984.

[*] Director of the scientific revue "MatematicaMente" and president of the Mathesis Verona; email: lcorso@iol.it

Il MIUR ha promosso ormai da anni il CLIL, acronimo di *Content and Language Integrated Learning*; una metodologia con cui si insegnano i contenuti disciplinari mediante l'uso di una lingua straniera.

Induzione, probabilità, corroborazione

Carlo Veronesi [*]

[*Segue dal numero* 242] Le teorie altamente corroborate sono, secondo Popper, più vicine alla verità delle teorie con un minor grado di corroborazione. In base alla concezione bayesiana della scienza potevamo parlare di teorie *probabilmente* vere. Popper ci dice che, in base alla sua teoria della corroborazione, si può pensare a teorie *approssimativamente* vere: anche se in futuro saranno confutate, contengono comunque una parte di verità.
Ma questa affermazione di Popper è piuttosto impegnativa. E su quali argomenti si basa? Non ci viene data una dimostrazione del legame tra la corroborazione e l'avvicinamento alla verità. Popper fonda il suo discorso sulla storia della fisica, in cui abbiamo visto che la teoria di Einstein è più vicina alla verità di quella di Newton, che a sua volta migliora le teorie di Copernico e Keplero, a loro volta più vicine alla verità dell'antico sistema tolemaico… Ma allora, se ricorriamo alla storia della scienza, vuol dire che pensiamo che il futuro sarà uguale al passato e ricadiamo ancora in una forma di induzione. L'induzione, che secondo Popper non è valida quando si applica a oggetti e osservazioni sulla natura, torna ad applicarsi a livello metateorico, quando parliamo, anziché di fenomeni naturali, delle teorie scientifiche che vertono su di essi. Popper ammette che può esserci "un pizzico di induttivismo in questo". I critici osservano che c'è molto di più: l'induzione, cacciata dalla porta, rientra dalla finestra.

RIFERIMENTI: [1] K.R. Popper, *Logica della scoperta scientifica*, Einaudi, Torino 1970, Appendice VII; [2] K.R. Popper, *Poscritto alla Logica della scoperta scientifica I. Il realismo e lo scopo della scienza*, Milano, il Saggiatore 1984; [3] K.R. Popper, *Repliche ai miei critici*, Rubbettino, Soveria Mannelli (CZ) 2016; [4] A. Einstein, *Induzione e deduzione nella fisica* (1919), tr. it. in «Nuova Civiltà delle Macchine», XIII, n. 49-50, 1995, pp. 149-150; [5] B. Russell, *I problemi della filosofia*, Feltrinelli, Milano 1980; [6] C. Dalla Pozza, *il problema della demarcazione. Verificabilità, falsificabilità e confermabilità bayesiana a confronto*, Editoria Scientifica Elettronica Università del Salento, Lecce 2008; [7] G. Boniolo - P. Vidali, *Introduzione alla filosofia della scienza*, Bruno Mondadori, Milano 2003.

L'autore ringrazia il prof. Pietro Baroni del Dipartimento di Ingegneria dell'Informazione dell'Università degli Studi di Brescia per le utili discussioni su alcuni punti del presente lavoro.

[*] Cultore di Logica e Filosofia della Scienza, Dipartimento di Scienze Umane - Università degli Studi di Verona